**SOLUTIONS**

## UBC Math104/184 Exam (December 2008)

**1.** (a) $\lim_{x\to 3} f(x) = \lim_{x\to 3}\frac{x^2-5x+6}{x^2-x-6} = \lim_{x\to 3}\frac{(x-3)(x-2)}{(x-3)(x+2)} = \lim_{x\to 3}\frac{x-2}{x+2} = \frac{3-2}{3+2} = \frac{1}{5}$.

If $f(x)$ is to be continuous at $x=3$, then we need $\lim_{x\to 3} f(x) = f(3)$. Since $f(3)=c$, then $c=\frac{1}{5}$.

(b) $$\lim_{h\to 0}\frac{\sqrt{4+h}-2}{h} = \lim_{h\to 0}\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} = \lim_{h\to 0}\frac{(4+h)-4}{h(\sqrt{4+h}+2)} = \lim_{h\to 0}\frac{h}{h(\sqrt{4+h}+2)}$$
$$= \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}$$

(c) $$\lim_{x\to\infty}\frac{x^3-5x+17}{7x^3+5x^2+x} = \lim_{x\to\infty}\frac{\frac{x^3}{x^3}-\frac{5x}{x^3}+\frac{17}{x^3}}{\frac{7x^3}{x^3}+\frac{5x^2}{x^3}+\frac{x}{x^3}} = \lim_{x\to\infty}\frac{1-\frac{5}{x^2}+\frac{17}{x^3}}{7+\frac{5}{x}+\frac{1}{x^2}} = \frac{1-0+0}{7+0+0} = \frac{1}{7}.$$

(d) $R'(q) = 3-0.2q+\frac{1}{0.5q}\cdot(0.5) = 3-0.2q+\frac{1}{q}$.

The marginal revenue is $R'(20) = 3-(0.2)\cdot 20+\frac{1}{20} = 3-4+0.05 = -0.95$.

(e) $$f'(x) = \frac{x\cdot\frac{1}{x}-\ln x\cdot 1}{x^2} = \frac{1-\ln x}{x^2}.$$

$$f''(x) = \frac{x^2\left(0-\frac{1}{x}\right)-(1-\ln x)\cdot 2x}{x^4} = \frac{-x-2x(1-\ln x)}{x^4} = \frac{-1-2(1-\ln x)}{x^3} = \frac{-1-2+2\ln x}{x^3} = \frac{2\ln x-3}{x^3}.$$

Therefore $f''(e) = \frac{2\ln e-3}{e^3} = \frac{2\cdot 1-3}{e^3} = -\frac{1}{e^3} < 0$, so $x=e$ a relative maximum.

(f) $A = Pe^{rt}$, where $A = 10{,}000$, $r = 4\% = 0.04$, and $t=7$. The amount to be invested is

$P = Ae^{-rt} = 10{,}000e^{(-0.04)\cdot 7} = 10{,}000e^{-0.28} = \$7557.84$.

(g) $f'(x) = x[e^{-x}\cdot(-1)]+e^{-x}\cdot 1 = -xe^{-x}+e^{-x} = (1-x)e^{-x}$.

$f(x) = xe^{-x}$ is decreasing when $f'(x)<0$, which occurs when $1-x<0$ (since $e^{-x}$ is always positive), i.e. when $x>1$. So $f(x)$ is decreasing on the interval $(1,\infty)$.

(h) $f'(x) = 3x^2-6x-9 = 3(x^2-2x-3) = 3(x-3)(x+1)$. At a critical point $f'(x)=0$, so the critical points are $x=3$ and $x=-1$. The only critical point on the interval $-2\le x\le 2$ is $x=-1$. There are also the two endpoints, $x=-2$ and $x=2$.

Since $f(-2) = (-2)^3-3(-2)^2-9\cdot(-2)+2 = -8-12+18+2 = 0$,

$f(-1) = (-1)^3 - 3(-1)^2 - 9\cdot(-1) + 2 = -1 - 3 + 9 + 2 = 7$,

and $f(2) = 2^3 - 3\cdot 2^2 - 9\cdot 2 + 2 = 8 - 12 - 18 + 2 = -20$,

the minimum value of the function $f(x) = x^3 - 3x^2 - 9x + 2$ on the interval $-2 \le x \le 2$ is $f(2) = -20$.

(i) $\dfrac{dy}{dx} = \dfrac{1}{1+(x^2)^2}\cdot 2x = \dfrac{2x}{1+x^4}$. The slope of the tangent line at the point $(1, \frac{\pi}{4})$ is $m = \left.\dfrac{dy}{dx}\right|_{x=1} = \dfrac{2\cdot 1}{1+1^4} = 1$.

So the equation of the tangent line at the point $(1, \frac{\pi}{4})$ is $y - \frac{\pi}{4} = 1(x-1)$ or $y = x + (\frac{\pi}{4} - 1)$.

(j) If $f(x)$ is to be differentiable at $x = 0$, it must first be continuous at $x = 0$,so

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \quad \Rightarrow \quad \lim_{x\to 0^-}(ax+b) = \lim_{x\to 0^+}(2\sin x + 3\cos x).$$

Therefore $a\cdot 0 + b = 2\sin 0 + 3\cos 0$ or $b = 2\cdot 0 + 3\cdot 1 = 3$. In addition, $f(x)$ must also satisfy

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x) \quad \Rightarrow \quad \lim_{x\to 0^-}(a) = \lim_{x\to 0^+}(2\cos x - 3\sin x).$$

Therefore $a = 2\cos 0 - 3\sin 0 = 2\cdot 1 - 3\cdot 0 = 2$. So $a = 2$ and $b = 3$.

(k) The value of the house after $t$ years is $A = Pe^{rt}$, where $P$ is the original value. When $t = 3$, $A = 2P$, so $2P = Pe^{r\cdot 3}$. Therefore, $e^{3r} = 2$, so $3r = \ln 2$ and $r = \dfrac{\ln 2}{3} \approx 0.2310 = 23.10\%$.

(l) $f(x) = \sqrt{x} = x^{1/2}$; $\qquad f(9) = \sqrt{9} = 3$;

$f'(x) = \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}}$; $\qquad f'(9) = \dfrac{1}{2\sqrt{9}} = \dfrac{1}{6}$;

$f''(x) = -\frac{1}{4}x^{-3/2} = -\dfrac{1}{4(\sqrt{x})^3}$; $\qquad f''(9) = -\dfrac{1}{4(\sqrt{9})^3} = -\dfrac{1}{108}$.

The second degree Taylor polynomial of $f(x)$ at $x = 9$ is

$$T_2(x) = f(9) + f'(9)(x-9) + \frac{f''(9)}{2!}(x-9)^2 = 3 + \tfrac{1}{6}(x-9) + \frac{-\frac{1}{108}}{2!}(x-9)^2 = 3 + \tfrac{1}{6}(x-9) - \tfrac{1}{216}(x-9)^2.$$

(m) $\dfrac{d}{dx}(x^2y^4) = \dfrac{d}{dx}(1) \quad \Rightarrow \quad x^2\cdot(4y^3y') + y^4\cdot(2x) = 0 \quad \Rightarrow \quad 4x^2y^3y' + 2xy^4 = 0 \quad \Rightarrow \quad y' = -\dfrac{2xy^4}{4x^2y^3} = -\dfrac{y}{2x}$.

So the slope of the tangent line at the point $(4, \frac{1}{2})$ is $m = y'|_{(4,\frac{1}{2})} = -\dfrac{\frac{1}{2}}{2\cdot 4} = -\dfrac{1}{16}$.

Since the tangent line passes through the point $(4, \frac{1}{2})$, its equation is $y - \frac{1}{2} = -\frac{1}{16}(x-4)$ or $y = -\frac{1}{16}x + \frac{3}{4}$.

(n) $\frac{d}{dt}(6p+q+qp)=\frac{d}{dt}(94) \quad \Rightarrow \quad 6\frac{dp}{dt}+\frac{dq}{dt}+\left(q\frac{dp}{dt}+p\frac{dq}{dt}\right)=0$.

Plugging in $p=9$, $q=4$, and $\frac{dp}{dt}=2$ gives

$$6\cdot 2+\frac{dq}{dt}+\left(4\cdot 2+9\frac{dq}{dt}\right)=0 \quad \Rightarrow \quad 12+\frac{dq}{dt}+8+9\frac{dq}{dt}=0 \quad \Rightarrow \quad 20+10\frac{dq}{dt}=0 \quad \Rightarrow \quad \frac{dq}{dt}=-2.$$

So demand is decreasing at a rate of 2 units per week.

**2.** (a) $f'(x)=\frac{(x^2+1)\cdot 2x-x^2\cdot 2x}{(x^2+1)^2} = \frac{2x[(x^2+1)-x^2]}{(x^2+1)^2} = \frac{2x}{(x^2+1)^2}$.

At a critical point $f'(x)=0$, so $\frac{2x}{(x^2+1)^2} = 0$. Therefore $x=0$.

So there is one critical point, $(0,f(0))=(0,0)$.

If $x<0$, then $f'(x)=\frac{2x}{(x^2+1)^2}<0$, so $f(x)$ is decreasing on the interval $(-\infty,0)$.

If $x>0$, then $f'(x)=\frac{2x}{(x^2+1)^2}>0$, so $f(x)$ is increasing on the interval $(0,\infty)$.

Therefore $(0,0)$ is a local minimum.

(b) At an inflection point $f''(x)=0$, so $\frac{2(1-3x^2)}{(x^2+1)^3}=0$. Therefore $1-3x^2=0$ or $x^2=\frac{1}{3}$. So there are two possible inflection points, namely $x=\frac{1}{\sqrt{3}}$ and $x=-\frac{1}{\sqrt{3}}$.

If $x<-\frac{1}{\sqrt{3}}$, then $f''(x)=\frac{2(1-3x^2)}{(x^2+1)^3}<0$, so $f(x)$ is concave down on the interval $(-\infty,-\frac{1}{\sqrt{3}})$.

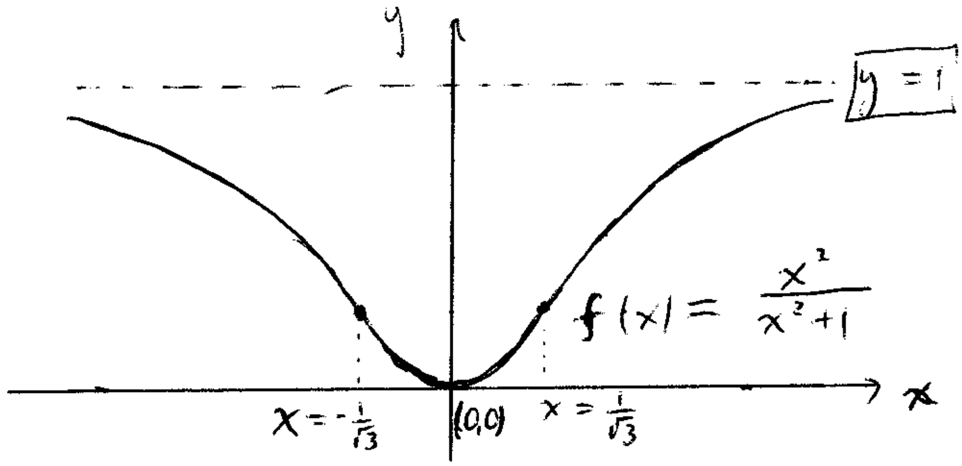
If $-\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}$, then $f''(x)=\frac{2(1-3x^2)}{(x^2+1)^3}>0$, so $f(x)$ is concave up on the interval $(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}})$.

If $x>\frac{1}{\sqrt{3}}$, then $f''(x)=\frac{2(1-3x^2)}{(x^2+1)^3}<0$, so $f(x)$ is concave down on the interval $(\frac{1}{\sqrt{3}},\infty)$.

So there really are inflection points at $x=\frac{1}{\sqrt{3}}$ and $x=-\frac{1}{\sqrt{3}}$ (since the concavity does change at these points).

(c) Since $\lim_{x\to\pm\infty} f(x)=\lim_{x\to\pm\infty}\frac{x^2}{x^2+1}=\lim_{x\to\pm\infty}\frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2}+\frac{1}{x^2}}=\lim_{x\to\pm\infty}\frac{1}{1+\frac{1}{x^2}} = \frac{1}{1+0} = 1$, therefore $y=1$ is a horizontal asymptote for $y=f(x)$. There are no vertical asymptotes since the denominator of $f(x)$ is never 0.

(d)




**3.** $\frac{dA}{dt} = 6 \cdot \frac{1}{2}(x^2-400)^{-1/2} \cdot 2x\frac{dx}{dt} = \frac{6x}{\sqrt{x^2-400}}\frac{dx}{dt}$.

Plugging in $x = 25$ and $\frac{dx}{dt} = 2$, gives

$$\frac{dA}{dt} = \frac{6 \cdot 25}{\sqrt{25^2-400}} \cdot 2 = \frac{300}{\sqrt{625-400}} = \frac{300}{\sqrt{225}} = \frac{300}{15} = 20.$$

So the advertising revenue is increasing at a rate of \$20,000 per month.

**4.** (a) $qp + 30p + 50q = 8500$, so $qp + 50q = 8500 - 30p$ or $q(p+50) = 8500 - 30p$, i.e. $q = \frac{8500-30p}{p+50}$.

So $q = f(p) = \frac{8500-30p}{p+50}$, and

$$f'(p) = \frac{(p+50)(-30)-(8500-30p)\cdot 1}{(p+50)^2} = \frac{-30p-1500-8500+30p}{(p+50)^2} = \frac{-10,000}{(p+50)^2}.$$

So the elasticity of demand is given by

$$E(p) = -\frac{pf'(p)}{f(p)} = -\frac{p \cdot \left[\frac{-10,000}{(p+50)^2}\right]}{\frac{8500-30p}{p+50}} = \frac{10,000p}{(p+50)^2} \cdot \frac{p+50}{8500-30p} = \frac{10,000p}{(p+50)(8500-30p)}.$$

When $p = 150$, $E(150) = \frac{10,000 \cdot 150}{(150+50)(8500-30 \cdot 150)} = \frac{1,500,000}{200 \cdot 4000} = \frac{1,500,000}{800,000} = \frac{15}{8} = 1.875.$

Since the elasticity is greater than 1, the revenue will decrease if $p$ increases and increase if $p$ decreases. Therefore revenue will increase if the price is lowered slightly.

(b) Revenue is $R = pq = pf(p) = p \cdot \dfrac{8500-30p}{p+50} = \dfrac{8500p-30p^2}{p+50}$. Therefore

$$\frac{dR}{dp} = \frac{(p+50)(8500-60p)-(8500p-30p^2)\cdot 1}{(p+50)^2}$$

$$= \frac{(8500p+425{,}000-60p^2-3000p)-8500p+30p^2}{(p+50)^2} = \frac{425{,}000-30p^2-3000p}{(p+50)^2}.$$

At a critical point $\dfrac{dR}{dp}=0$, so $425{,}000-30p^2-3000p=0$ or $42{,}500-3p^2-300p=0$.

Therefore $3p^2+300p-42{,}500=0$, so

$$p = \frac{-300 \pm \sqrt{300^2-4\cdot 3\cdot(-42{,}500)}}{2\cdot 3} = \frac{-300\pm\sqrt{90{,}000+510{,}000}}{6} = \frac{-300\pm\sqrt{600{,}000}}{6}$$

$$= \frac{-300\pm\sqrt{40{,}000\cdot 15}}{6} = \frac{-300\pm 200\sqrt{15}}{6} = \frac{-300}{6} \pm \frac{200\sqrt{15}}{6} = -50 \pm \tfrac{100}{3}\sqrt{15}.$$

Choosing the positive value gives $p = -50+\frac{100}{3}\sqrt{15} \approx 79.10$. The phone supplier should set the price to be \$79.10.

**5.** Let $x$ be the number of orders made during the year. Since 640 sofas will be sold during the year, each order is for $N = \dfrac{640}{x}$ sofas. The average number of sofas in inventory throughout the year is half of that, i.e. $\frac{1}{2}N = \dfrac{320}{x}$. The total cost is therefore

$$C(x) = 160x + 32\cdot(\tfrac{1}{2}N) = 160x+16N = 160x+16\cdot\frac{640}{x} = 160(x+64x^{-1}).$$

At a critical point $C'(x) = 160(1-64x^{-2}) = 0$, so $1 = 64x^{-2} = \dfrac{64}{x^2}$ or $x^2=64$. Therefore $x=8$ (since $x>0$).

So they should make 8 orders per year, and each order should be for $N = \dfrac{640}{8} = 80$ sofas.

**6.** Setting $P=1000$, $R=220$, and $N=5$, gives $1000i+220[(1+i)^{-5}-1]=0$.

Let $f(x)=1000x+220[(1+x)^{-5}-1]$. Then $f'(x)=1000+220[-5(1+x)^{-6}-0] = 1000-1100(1+x)^{-6}$.

Starting with the initial guess $x_0=0.03$ gives

$$x_1 = x_0-\frac{f(x_0)}{f'(x_0)} = 0.03-\frac{f(0.03)}{f'(0.03)} = 0.03-\frac{1000\cdot(0.03)+220[(1.03)^{-5}-1]}{1000-1100(1.03)^{-6}}$$

$$= 0.03-\frac{-0.2260674355}{78.76731765} = 0.0328700664$$

$$x_2 = x_1-\frac{f(x_1)}{f'(x_1)} = 0.0328700664-\frac{f(0.0328700664)}{f'(0.0328700664)}$$

$$= 0.0328700664-\frac{0.0219593827}{94.02015309} = 0.0326365061.$$

The monthly rate of interest is approximately 3.26%.